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Noise effects on one-Pauli channels

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Abstract. The possibility of stochastic resonance of a quantum channel and hence the noise enhanced quantum channel capacity is explored by considering one-Pauli channels which are more classical like. The fidelity of the channel is also considered.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 03.67.Hk Quantum communication

1 Introduction

Recently because of the development of quantum computers [1] people have become interested in information transmission through quantum channels [2]. Quantum information theories [3] can be used to describe processes such as data storage, quantum cryptography [4] and quantum teleportation [5]. However, after an initial burst of papers following Shor's discover of quantum factoring algorithm [6], almost every work is aiming to solve the decoherence problem which is responsible for transition into effectively classical behaviour [7]. There are people using NMR techniques, which provide longer decoherence time than previous techniques, claiming they can built a quantum computer with a cup of coffee [8]. There are also people trying to use various software methods, in particular, quantum error correcting codes, to correct decoherence induced errors. The decoherence problem lay at the heart of the development of quantum mechanics. Apparently, decoherence is a hurdle need to be surmounted before quantum computers can be materialized. However, is decoherence, the counterpart of classical noise, really nuisance? For people who know stochastic resonance [9], the answer is perhaps "no". Decoherence can be perhaps used as a resource as entanglement had.

In two previous papers we have considered the noise effects on the two-Pauli channels [10] and the depolarizing channels [11] using the concept from stochastic resonance. There is a question that whether does the results obtained so far consistent with the classical results? For the two-Pauli channel and depolarizing channel, it is not possible to check because they consist of flipping the phase of the qubits (quantum bits) [12] which has no classical counter part. However, it might be possible to consider a one-Pauli channel that only flip the qubits amplitude. The σ_1 channel can be viewed as a binary symmetric channel, while σ_2 and σ_3 channels are σ_1 channel in a dual basis. What are the noise effects for each Pauli operator separately? In this paper we will consider three different one-Pauli channels to see how will the noise influence their capacity and fidelity.

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2 The noisy channel model

The classical world is made from different material. However, in the quantum world all objects are made of the same elementary particles. The particles are in different states of superposition. Only the information describing them are different.

In the classical world the information is coded as *bits* and is described by 0 or 1, while the quantum world the information is coded as *qubit* and is described by the corresponding *density matrix*.

Schumacher and Nielsen [13] have developed a quantum information theory to describe the information processing in the quantum world. In their formulation a quantum channel can be considered as a process defined by an input density matrix ρ_x, and an output density matrix ρ_y, with the process described by a quantum operation, $\mathcal{N}$,

$$\rho_x \xrightarrow{\mathcal{N}} \rho_y. \tag{1}$$

Because of decoherence, the super-operator $\mathcal{N}$ is not unitary. However, on a larger quantum system that includes the environment E of the system, the total evolution operator U_{xE} become unitary. This environment may be considered to be initially in a pure state $|0_E\rangle$ without loss of generality. In this case, the super-operator can be written as

$$\mathcal{N}(\rho_x) = \mathrm{Tr}_E U_{xE} \left(\rho_x \otimes |0_E\rangle\langle 0_E|\right) U_{xE}{}^{\dagger}. \tag{2}$$

The partial trace, Tr_E, is taken over environmental degree of freedom, and $\otimes$ means a direct product for the spaces on both sides of the operator. Equation (2) can be rewritten as a completely positive linear transformation acting on the density matrix:

$$\mathcal{N}(\rho_x) = \sum_i A_i \rho_x A_i^\dagger, \qquad (3)$$

in which the A_i satisfy the completeness relation

$$\sum_i A_i^\dagger A_i = I, \qquad (4)$$

which is equivalent to requiring $\mathrm{Tr}[\mathcal{N}(\rho_x)] = 1$. The mutual information of a classical channel with classical sources, written using quantum formalism become [14]

$$H(x:y) = H(\rho_x) + H(\mathcal{N}(\rho_x)) - H_e(\rho_x, \mathcal{N}), \qquad (5)$$

in which $H(\bullet) = -\mathrm{Tr}\left[\bullet \log_2 \bullet\right]$ is the von Neumann entropy [15], and

$$H_e(\rho_x, \mathcal{N}) \equiv -\mathrm{Tr}(W \log_2 W), \qquad (6)$$

with

$$W_{ij} \equiv \mathrm{Tr}(A_i \rho_x A_j^\dagger) \qquad (7)$$

measures the amount of information exchanged between the system x and the environment E during their interaction [2], which can be used to characterize the amount of quantum noise, N, in the channel. If the environment is initially in a pure state, the entropy exchange is just the environment's entropy after the interaction. Hence, the coherent information is defined as

$$C(\rho_x, \mathcal{N}) \equiv H\left(\mathcal{N}(\rho_x)\right) - N(\rho_x, \mathcal{N}), \qquad (8)$$

which plays a role in quantum information theory analogous to that played by the mutual information in classical information theory.

3 One-Pauli channels

In what follows the influence of noise on three kinds of one-Pauli channels with a general input state

$$\rho_x = \frac{1}{2}\left(I + \mathbf{a}\cdot\boldsymbol{\sigma}\right) \qquad (9)$$

is considered. Here, I is the identity matrix, $\mathbf{a} = (a_1, a_2, a_3)$ is the Bloch vector of length unity or less, and $\boldsymbol{\sigma}$ is the vector of Pauli matrices, which are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (10)$$

The output of the channel can always be written as

$$\mathcal{N}(\rho_x) = \frac{1}{2}\left(I + \mathbf{b}\cdot\boldsymbol{\sigma}\right). \qquad (11)$$

Two more definitions are needed in our computing for the channel properties: The von Neumann entropy is

$$H(\rho_x) = -\sum_i \theta_i \log \theta_i, \qquad (12)$$

in which θ_is are the eigenvalues of the density matrix ρ_x. Furthermore, the (entangled) fidelity

$$F = \sum_\mu (\mathrm{Tr}\rho_x A_\mu)(\mathrm{Tr}\rho_x A_\mu^\dagger), \qquad (13)$$

is also of our concern, since it represent the quality of the signal transmitted.

3.1 σ_1 channel

A σ_1 channel can be written in terms of A_i's in equation (3) as

$$A_1 = \sqrt{x} I, \quad A_2 = \sqrt{1-x}\,\sigma_1. \qquad (14)$$

This channel flips the qubit amplitude with probability $1-x$. We have

$$\boldsymbol{b} = \Big(a_1,\, a_2(2x-1),\, a_3(2x-1)\Big). \qquad (15)$$

The matrix W for the σ_1 channel read,

$$W = \begin{pmatrix} x & a_1\sqrt{x(1-x)} \\ a_1\sqrt{x(1-x)} & 1-x \end{pmatrix}. \qquad (16)$$

It eigenvalues are

$$\lambda_{1,2} = \left[1 \pm \sqrt{1 - 4x(x-1)(a_1^2 - 1)}\right]/2. \qquad (17)$$

Hence,

$$N = -\sum_{i=1}^{2} \lambda_i \log_2 \lambda_i, \qquad (18)$$

while $\theta_{1,2} = \left[1 \pm \sqrt{a_1^2 + (a_2^2 + a_3^2)(1-2x)^2}\right]/2$. This $x - N$, $C - N$ relationship is plotted in Figure 1a.

For the σ_1 channel the entangled fidelity

$$F = a_1^2(1-x) + x. \qquad (19)$$

The relation between the fidelity and the noise is plotted with the coherent information in Figure 1a.

3.2 σ_2 channel

Similarly, a σ_2 channel can be written in terms of A_i's in equation (3) as

$$A_1 = \sqrt{x} I, \quad A_2 = -i\sqrt{1-x}\,\sigma_2. \qquad (20)$$

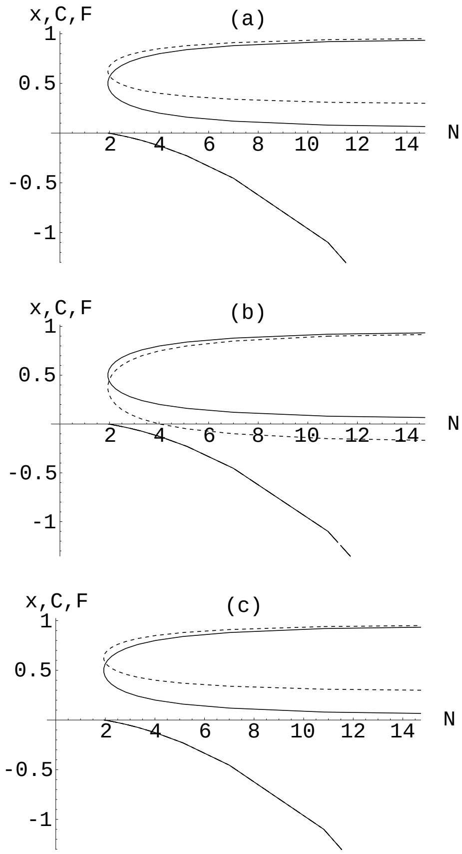

Fig. 1. Parametric plots of the retention rate, x, *versus* noise, N (solid lines); coherent information, C, *versus* noise, N, (long dashed lines) and fidelity, F, *versus* noise, N, (short dashed lines) for the parameter x from 0 to 1 and (a) σ_1 channel with initial state $a_1 = 5/10, a_2 = 6/10, a_3 = 6/10$; (b) σ_2 channel with initial state $a_1 = 6/10, a_2 = 5/10, a_3 = 6/10$; (c) σ_3 channel with initial state $a_1 = 6/10, a_2 = 6/10, a_3 = 5/10$.

This channel flips the qubit amplitude and phase with probability $1-x$. The action of the channel on this density matrix is:

$$\mathcal{N}(\rho_x) = \frac{1}{2}\Big(I + \mathbf{b}\cdot\boldsymbol{\sigma}\Big), \tag{21}$$

in which

$$\mathbf{b} = \Big(a_1(2x-1),\, a_2,\, a_3(2x-1)\Big)\,. \tag{22}$$

The matrix W for the σ_2 channel read,

$$W = \begin{pmatrix} x & \mathrm{i}a_2\sqrt{x(1-x)} \\ -\mathrm{i}a_2\sqrt{x(1-x)} & 1-x \end{pmatrix}. \tag{23}$$

It eigenvalues are

$$\lambda_{1,2} = \left[1 \pm \sqrt{1-4x(x-1)(a_2^2-1)}\right]/2. \tag{24}$$

Hence,

$$N = -\sum_{i=1}^{2} \lambda_i \log_2 \lambda_i, \tag{25}$$

while $\theta_{1,2} = \left[1 \pm \sqrt{a_2^2 + (a_1^2+a_3^2)(1-2x)^2}\right]/2$. For the σ_2 channel

$$F = -a_2^2(1-x) + x. \tag{26}$$

The relation between the fidelity and the noise is plotted with the coherent information in Figure 1b.

3.3 σ_3 channel

A σ_3 channel can be written as

$$A_1 = \sqrt{x}I, \quad A_2 = \sqrt{1-x}\sigma_3. \tag{27}$$

This channel flips the qubit phase with probability $1-x$. The action of the channel on this density matrix is:

$$\mathcal{N}(\rho_x) = \frac{1}{2}\Big(I + \mathbf{b}\cdot\boldsymbol{\sigma}\Big), \tag{28}$$

in which

$$\mathbf{b} = \Big(a_1(2x-1),\, a_2(2x-1),\, a_3\Big). \tag{29}$$

The matrix W for the σ_3 channel read,

$$W = \begin{pmatrix} x & a_3\sqrt{x(1-x)} \\ a_3\sqrt{x(1-x)} & 1-x \end{pmatrix}. \tag{30}$$

It eigenvalues are

$$\lambda_{1,2} = \left[1 \pm \sqrt{1-4x(x-1)(a_3^2-1)}\right]/2. \tag{31}$$

Hence,

$$N = -\sum_{i=1}^{2} \lambda_i \log_2 \lambda_i, \tag{32}$$

while $\theta_{1,2} = \left[1 \pm \sqrt{a_3^2 + (a_1^2+a_2^2)(1-2x)^2}\right]/2$. For the σ_3 channel

$$F = a_3^2(1-x) + x. \tag{33}$$

The relation between the fidelity and the noise is plotted with the coherent information in Figure 1c.

We noticed that for all three channels the capacity is always non-positive whatever the amount of noise is.

4 Conclusion

In conclusion we found, as far as the coherent information and fidelity are concerned, these three one-Pauli channels are the same except for the fidelity of the σ_2 channel. This is perhaps the reason that we found in the previous works why there is noise enhancement of the fidelity but could not find noise enhancement for the coherent information.

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